Creation of Spin-1/2 Particles in the Hyperboloid de Sitter Space-Time

Ali Havare, Taylan Yetkin, Murat Korunur
Department of Physics, Mersin University, 33342, Mersin-Turkey

February 7, 2008

Abstract

In this work we solve Dirac equation by using the method of seperation of variables. Then we analyzed the particle creation process. To compute the density number of particles created Bogoliubov transformation technique is used.

1 Introduction

The creation of elementary particles in curved space-time by strong fields is one of the most exciting problem and still matter of investigation in contemporary theoretical physics. In the literature a lot of studies can be found concerning with the phenomenon of particle creation, mainly in four-dimensional isotropic and homogeneous cosmological models as well as two-dimensional toy models[1, 2, 3, 4]. In curved space-time quantum field vacuum must be defined carefully since the vacuum plays an important role in understanding the nature of particle creation. One of the most studied technique to compute the densitiy number of particles created is the Bogoliubov Transformation Technique(BTT)[5]. In this technique 'in' and 'out' vacuum states are determined from the solutions of relativistic wave equations using a particular choice of time. Then the vacuum is specified $|in, 0\rangle$ and $|out, 0\rangle$, and the Bogoliubov mixing transformation is found between them;

$$\Psi_{in,k}(x) = \alpha\Psi_{out,k}(x) + \beta\Psi^{*}_{out,-k}(x), \tag{1}$$

where α and β are mixing coefficients, and they are related to the density number of particles created. The studies in the literature have considered four-dimensional isotropic space-times as well as two-dimensional simple models.

In this work we will use two-dimensional de Sitter space-time represented as the hyperboloid

$$z_0^2 - z_1^2 - z_2^2 = -a^2 \tag{2}$$

embeded in three dimensional Minkowski space[6], where

$$z_0 = a\sinh(t/a), \tag{3}$$

$$z_1 = a\cosh(t/a)\cos x, \tag{4}$$

$$z_2 = \cosh(t/a)\sin x. \tag{5}$$

The line element of the space-time is given by

$$ds^2 = -dt^2 + a^2\cosh^2(t/a)dx^2, (-\pi < x < \pi). \tag{6}$$

Because of the simple form of the line element the solution of the Dirac equation and definition of Bogoliubov coefficients are easy. For the model in Eq.(5) the Hawking effect for scalar particles has been studied by using BTT[7].

2 Solution of the Dirac Equation

The covariant generalization of Dirac equation is

$$[i\gamma^\mu(\partial_\mu - \Gamma_\mu) - m]\,\Psi(x) = 0, \tag{7}$$

where γ^μ are curved space-time Dirac gamma mattricies related to flat space-time gammas as $\gamma^\mu(x) = e^\mu_{(i)}\gamma^{(i)}$ with the tetrad definition

$$e^\mu_{(i)}e^\nu_{(k)}\eta^{(i)(k)} = g^{\mu\nu}. \tag{8}$$

The spin connections Γ_λ are defined

$$\Gamma_\lambda = -\frac{1}{8}g_{\mu\alpha}\Gamma^\alpha_{\nu\lambda}[\gamma^\mu, \gamma^\nu], \tag{9}$$

where $\Gamma^\alpha_{\nu\lambda}$ are Christoffel symbols given

$$\Gamma^\alpha_{\nu\lambda} = \frac{1}{2}g^{\alpha\beta}(\partial_\nu g_{\lambda\beta} + \partial_\lambda g_{\beta\lambda} - \partial_\beta g_{\lambda\nu}). \tag{10}$$

We can simplify calculations by introducing a conformal time given

$$\eta = 2\arctan[\exp(t/a)], \tag{11}$$

then Eq.(6) becomes

$$ds^2 = \frac{a^2}{\sin^2\eta}(-d\eta^2 + dx^2), (0 < \eta < \pi). \tag{12}$$

The Dirac equation Eq.(7), after multiplying $-\gamma^{(0)}$ from left, becomes

$$(\partial_\eta - \gamma^{(0)}\gamma^{(1)}ik - \frac{ima}{\sin\eta}\gamma^{(0)})\Phi(\eta) = 0, \tag{13}$$

where the function of the form

$$\Psi = \frac{1}{\sqrt{\sin x}} e^{ikx} \Phi(\eta) \tag{14}$$

introduced to separate variables and to cancel out the terms coming from spin connections. If we choose Dirac matrices in two dimensions as ($\{\gamma^{(i)}, \gamma^{(k)}\} = 2\eta^{(i)(k)}$)

$$\gamma^{(0)} = i\sigma^3, \gamma^{(1)} = \sigma^1, \tag{15}$$

Eq.(13) gives two coupled equations

$$(\partial_\eta + \frac{ma}{\sin\eta})\Phi_1(\eta) - k\Phi_2(\eta) \quad = \quad 0, \tag{16}$$

$$(\partial_\eta - \frac{ma}{\sin\eta})\Phi_2(\eta) + k\Phi_1(\eta) \quad = \quad 0, \tag{17}$$

It is easy to see that each component of the spinor satisfies hypergeometric differential equation if we introduce the following functions in Eqs.(16) and (17)

$$\Phi_1(\eta) \quad = \quad \sin^{ma}\eta \sin\frac{\eta}{2} f_1(\eta), \tag{18}$$

$$\Phi_2(\eta) \quad = \quad \sin^{ma}\eta \cos\frac{\eta}{2} f_2(\eta). \tag{19}$$

Then we obtain

$$[(x-1)\partial_x + \alpha]f_1 + kf_2, \quad = \quad 0, \tag{20}$$

$$[(x+1)\partial_x + \alpha]f_2 + kf_1, \quad = \quad 0, \tag{21}$$

where

$$\alpha = ma + 1/2; \;\; x = \cos\eta. \tag{22}$$

Substituting Eq.(20) into Eq.(21) we obtain

$$\left\{u(1-u)\partial_u^2 + [C - (A+B+1)u]\partial_u - AB\right\} f_1(u) = 0, \tag{23}$$

where

$$u = \frac{x+1}{2}; A_{1,2} = ma + \frac{1}{2} \pm k; B_{1,2} = ma + \frac{1}{2} \mp k; C = ma + \frac{1}{2}. \tag{24}$$

Solution to Eq.(23) is[8]

$$f_1(x) = N\, {}_2F_1(ma + \frac{1}{2} \pm k, ma + \frac{1}{2} \mp k, ma + \frac{1}{2}; \frac{x+1}{2})$$

$$+ M\left(\frac{x+1}{2}\right)^{\frac{1}{2}-ma} {}_2F_1(\pm k + 1, \mp k + 1, -ma - \frac{3}{2}; \frac{x+1}{2}), \tag{25}$$

where N and M are normalization constants. The second component of the spinor can be found using the recursion relations of the hypergeometric functions. Then the exact solution of the first component of the spinor is found as follow:

$$\Psi_1(\eta,x)=e^{ikx}\sin^{ma-\frac{1}{2}}\eta\sin\frac{\eta}{2}[N\,{}_2F_1(ma+\frac{1}{2}\pm k,ma+\frac{1}{2}\mp k,ma+\frac{1}{2};\frac{\cos\eta+1}{2})$$

$$+M\left(\frac{\cos\eta+1}{2}\right)^{\frac{1}{2}-ma}{}_2F_1(\pm k+1,\mp k+1,-ma-\frac{3}{2};\frac{\cos\eta+1}{2})]. \quad (26)$$

3 Particle Creation

In order to analyze the phenomenon of the particle creation we proceed to discuss the behaviour of the solutions of the Dirac equation in the hypersurfaces $\eta=0,\pi$ or equivalently when $x=\pm1$. To relate two different vacuum for $x=\pm1$, we can make use of the relation for the hypergeometric functions,

$${}_2F_1(\alpha,\beta,\gamma;z)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,{}_2F_1(\alpha,\beta,\alpha+\beta-\gamma+1;1-z)$$

$$+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\,{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z), \quad (27)$$

then the negative frequency mode for $x=-1$ reads

$$f^-_{(x=-1)}=C^-\,{}_2F_1(ma+\frac{1}{2}\pm k,ma+\frac{1}{2}\mp k,ma+\frac{1}{2};\frac{x+1}{2}) \quad (28)$$

whereas the positive frequency mode for $x=+1$ takes the form

$$f^+_{(x=+1)}=C^+\,{}_2F_1(ma+\frac{1}{2}\pm k,ma+\frac{1}{2}\mp k,ma+\frac{1}{2};\frac{x+1}{2}), \quad (29)$$

where C^+ and C^- are normalization constants. Using the relation (**??**) we obtain

$$f^-_{(x=-1)}=\frac{\Gamma(ma+\frac{1}{2})\Gamma(-ma-\frac{1}{2})}{\Gamma(k)\Gamma(-k)}f^+_{(x=+1)}+\frac{\Gamma(ma+\frac{1}{2})\Gamma(-ma-\frac{1}{2})}{\Gamma(ma+\frac{1}{2}+k)\Gamma(ma+\frac{1}{2}-k)}f^-_{(x=+1)}. \quad (30)$$

Finally we obtain using Bogoliubov coefficients

$$n\sim\frac{|\beta|^2}{|\alpha|^2}=\left|\frac{\Gamma(ma+\frac{1}{2}+k)\Gamma(ma+\frac{1}{2}-k)}{\Gamma(k)\Gamma(-k)}\right|^2 \quad (31)$$

From Eq.(31) and relation $|\alpha|^2+|\beta|^2 = 1$ we find that the density of particles created $|\beta|^2$ is given

$$|\beta|^2 = \frac{\sin^2 k\pi}{\cos^2(ma + \frac{1}{2} + k)\pi}. \tag{32}$$

The expression (**??**) shows that the density of particles created has a maximum for $ma + k = n - \frac{1}{2}$, where n is integer. Also have that there is no particle creation when k takes integer values. Hence, if the universe has the phenomenon of particle creation, the momentum can not take integer values. When massless particles couples conformally to the metric [3] Eq.(32) takes the form

$$|\beta|^2 = \frac{\sin^2 k\pi}{\cos^2(\frac{1}{2} + k)\pi} = 1. \tag{33}$$

Equation (33) shows that the density of particles created has a maximum for negative modes.

For the de Sitter space-time studied by Villalba[9] the density of particles created is

$$|\beta|^2 = \frac{1}{e^{2\pi m/H} + 1}. \tag{34}$$

We can see that both results in Eqs.(32) and (34) are depend on the mass value of the particle while in Eq.(32) it also depends on momentum value of the particle.

4 Conclusions

We solved the Dirac equation using the merhod of seperation of variables for de Sitter space-time. Then using the BTT method we computed that the density number of particles created and also found that the particle creation does not occur when the momentum of spin-1/2 particle has integer values.

References

[1] S. Biswas, J. Guha and G. Sarkar, Class. Quantum Grav., **12**,1591(1995).

[2] J. Audretsch, Gen. Rel. Grav., **10**,725(1979).

[3] V.M. Villaba, Phys.Rev. D52, 3742(1995).

[4] A.A. Grib, Gen.Rel. and Grav., **32**,621(2000).

[5] N.D. Birrel and P.C.W. Davies, *Quantum fields in Curved Space*, Cambridge Universty Press(1982).

[6] E. Schrödinger, *Expanding Universes*, Cambridge University Press, Cambridge, England (1956).

[7] T. Mishima and A. Nakayama, Phys. Rev.**D37**, 348(1988).

[8] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1974).

[9] Ya. B. Zeldovich and A.A. Starobinskii, Sov. Phys. JETP**61**, 2161(1971), V.M. Villalba, arXiv:gr-qc/9410026v/.